

# A Few Finite Trigonometric Sums

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## Abstract

Finite trigonometric sums occur in various branches of physics, mathematics, and their applications. These sums may contain various powers of one or more trigonometric functions. Sums with one trigonometric function are known, however sums with products of trigonometric functions can get complicated and may not have a simple expressions in a number of cases. Some of these sums have interesting properties and can have amazingly simple value. However, only some of them are available in literature. We obtain a number of such sums using method of residues.

## 1 Introduction

There is a venerable tradition of computing finite sums of product of trigonometric functions in literature [1]. Such sums occur while addressing many different problems in physics, or mathematics, or their applications. Sums such as [2]

$$\sum_{j=1}^{d-1} \sin \frac{2\pi m j}{d} \cot\left(\frac{\pi j}{d}\right) = k - 2m, \quad (1)$$

are known for a long time. However, if a small variations of arguments of these functions is made, like arguments are affine functions, then these sums no longer remain easy to compute and are not available in various standard handbooks of mathematics [3], including those specialize in series sum [4, 5, 6]. As an example, we may wish to compute

$$\sum_{j=1}^{d-1} \sin\left(\frac{2\pi m j}{d} + a\right) \cot\left(\frac{\pi j}{d} + b\right). \quad (2)$$

There exist useful sum and difference formulas for sines and cosines that can be used, but such formulate don't exist for other trigonometric functions. In such cases, there is a need to compute sums separately. For example, the above sum can be computed for  $b = 0$ , using existing results in the handbooks, but for  $b \neq 0$ , the sum is nontrivial.

We have encountered such a sum in computing the violation of a Bell-type inequality for a system of two finite-dimensional subsystems. We encountered the following sum [7],

$$\sum_{\alpha=0}^{d-1} \cos\left(\frac{2\pi m}{d}\left(\alpha + \frac{1}{4}\right)\right) \cot\left(\frac{\pi}{d}\left(\alpha + \frac{1}{4}\right)\right). \quad (3)$$

We can use cosine sum rule, but there is no such rule for cotangent. So we cannot compute it using results given in standard handbooks. This sum can be computed using a corollary, given below. Interestingly, this complicated

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looking sum has the value  $d$ , which is remarkably simple, and is independent of  $m$ . Independence on  $m$  is intriguing, and may have deeper mathematical meaning.

There are many techniques for computing these sums, e.g., use of generating functions [8], Fourier analysis, method of residues, etc. Sometimes, the same sum can be obtained by different methods, giving different looking results. In this work, we will employ the method of residues [9, 10]. We will start with a function with suitable singularity structure. The function and contour will be chosen in such a way that integration of the function over the contour gives the desired series and its sum. So the main trick is to find suitable functions and computing residues at poles.

In the next section, we have computed a few sums involving product of two different trigonometric functions. In Section 3, we generalize the results to the sums of the product of more than two trigonometric functions. In Section 4, we conclude.

## 2 Products of Two Trigonometric Functions

In this section, we will compute the finite sums involving the product of sine with powers of cotangent/cosecants and cosine with powers of cotangent/cosecants. As a byproduct, we will also get a result involving tangent instead of cotangent. In computing these sums, we will use the expansion

$$\frac{1}{te^z - 1} = \sum_{\nu=0}^{\infty} \frac{A_{\nu}(t)}{\nu!} z^{\nu}, \quad (4)$$

where  $A_{\nu}(t)$  is a polynomial in  $t$  of degree  $\nu$ . The first few polynomials are  $A_0(t) = \frac{1}{t-1}$ ,  $A_1(t) = \frac{-t}{(t-1)^2}$ ,  $A_2(t) = \frac{t+t^2}{(t-1)^3}$  and  $A_3(t) = \frac{-(t+4t^2+t^3)}{(t-1)^4}$ . We will also need to expand cotangent in a power series

$$\cot(\pi z + \pi b) = \sum_{j=0}^{\infty} C_j \pi^{2j-1} (z - 1 + b)^{2j-1}, \quad (5)$$

where  $C_j$  is a polynomial of degree  $j$ . The first few terms are  $C_0 = 1$ ,  $C_1 = -\frac{1}{3}$ ,  $C_2 = -\frac{1}{45}$  and  $C_3 = -\frac{2}{945}$ .

Let us start with the following theorem

**Theorem 1.1 :**

$$\begin{aligned} e_n(d, m) &= \sum_{j=0}^{d-1} \cos \frac{2\pi m j}{d} \cot^n \left( \frac{\pi j}{d} + \pi b \right) \\ &= - \sum_{D(j_1, j_2, \dots, j_n)} i^{\mu+\nu+1} 2^{\mu+\nu} \frac{m^{\mu}}{\mu!} \frac{d^{\nu+1}}{\nu!} \left( t_1 A_{\nu}(t_2) - (-1)^{\mu+\nu} t_1' A_{\nu}(t_2') \right) \end{aligned} \quad (6)$$

where  $m, n$  and  $d$  denote positive integers with  $m < d$ . Here the sum is over all nonnegative integers  $j_1, \dots, j_n, \mu$  and  $\nu$  such that  $2j_1 + \dots + 2j_n + \mu + \nu = n-1$ . We also have  $t_1 = e^{-2\pi i m(b-1)}$ ,  $t_2 = e^{-2\pi i d(b-1)}$ ,  $t_1' = e^{2\pi i m(b-1)}$  and  $t_2' = e^{2\pi i d(b-1)}$ , with  $b$  in between 0 and 1. Furthermore,

$$D(j_1, j_2, \dots, j_n) = \prod_{r=1}^n C_{j_r}. \quad (7)$$

**Proof :** We choose contour  $C_R$  as a positively oriented indented rectangle with vertices at  $\pm iR$  and  $1 \pm iR$ . The contour has two semicircular indentations of radius  $\epsilon$  ( $R > \epsilon$ ) to the left of both 0 and 1 [9]. Let us take the complex function as

$$f(z) = \frac{e^{2\pi i m z} \cot^n(\pi z + \pi b)}{e^{2\pi i d z} - 1} - \frac{e^{-2\pi i m z} \cot^n(\pi z + \pi b)}{e^{-2\pi i d z} - 1} \quad (8)$$

and consider  $\frac{1}{2\pi i} \int_C f(z) dz$ . Since  $f(z)$  has period 1, the integrals along the indented vertical sides of  $C_R$  cancel. Since we have taken  $m < d$ ,  $f(z)$  tends to zero uniformly for  $0 \leq x \leq 1$  as  $|y| \rightarrow \infty$ . Hence,  $\frac{1}{2\pi i} \int_C f(z) dz = 0$ . We can now calculate the contour integral using Cauchy's residue theorem. The function  $f(z)$  has poles at a number of points. To start with,  $f(z)$  has a simple pole at  $z = 0$ , with residue

$$\text{Res}(f, 0) = \frac{1}{\pi i d} \cot^n(b\pi). \quad (9)$$

The function  $f(z)$  also has simple poles at  $z = \frac{j}{d}$ , with  $1 \leq j \leq d-1$ . The corresponding residues at these points are

$$\text{Res}(f, \frac{j}{d}) = \frac{1}{\pi i d} \cos \frac{2\pi m j}{d} \cot^n(\frac{\pi j}{d} + \pi b). \quad (10)$$

In addition the function  $f(z)$  has a pole of order  $n$  at  $z = -b + 1$ . Using equations (4) and (5), we can write

$$\begin{aligned} f(z) &= t_1 \sum_{\mu=0}^{\infty} \frac{(2\pi i m)^\mu}{\mu!} (z+b-1)^\mu \left( \sum_{j=0}^{\infty} C_j \pi^{2j-1} (z-1+b)^{2j-1} \right)^n \\ &\quad \sum_{\nu=0}^{\infty} \frac{(2\pi i d)^\nu}{\nu!} A_\nu(t_2) (z+b-1)^\nu \\ &\quad - t'_1 \sum_{\mu=0}^{\infty} \frac{(-2\pi i m)^\mu}{\mu!} (z+b-1)^\mu \left( \sum_{j=0}^{\infty} C_j \pi^{2j-1} (z-1+b)^{2j-1} \right)^n \\ &\quad \sum_{\nu=0}^{\infty} \frac{(-2\pi i d)^\nu}{\nu!} A_\nu(t'_2) (z+b-1)^\nu. \end{aligned} \quad (11)$$

Then after few steps of straightforward calculation, one can show that,

$$\text{Res}(f, -b+1) = \sum i^{\mu+\nu} 2^{\mu+\nu} \frac{m^\mu}{\mu!} \frac{d^\nu}{\nu!} \left( \frac{t_1}{\pi} A_\nu(t_2) - (-1)^{\mu+\nu} \frac{t'_1}{\pi} A_\nu(t'_2) \right) D(j_1, j_2, \dots, j_n). \quad (12)$$

Here the sum is over all nonnegative integers  $j_1, \dots, j_n$ ,  $\mu$  and  $\nu$  such that  $2j_1 + \dots + 2j_n + \mu + \nu = n-1$ . Using (9), (10), (12) and applying residue theorem we can obtain the sum (6).

**Corollary 1.2 :** Let  $m$  and  $d$  be the integers such that  $0 < m < d$ . Then

$$e_1(d, m) = d \cos[(2m-d)b\pi] \text{cosec}(bd\pi). \quad (13)$$

**Proof :** Put  $n = 1$  in Theorem 1.1. Using the values  $A_0(t)$  and  $C_0$ , one can easily see this.

We will now consider sums involving sine and cotangents. We will need to modify our functions suitably.

**Theorem 1.3 :** Let  $m, n$  and  $d$  denote positive integers with  $m < d$ . Then

$$\begin{aligned} g_n(d, m) &= - \sum i^{\mu+\nu} 2^{\mu+\nu} \frac{m^\mu}{\mu!} \frac{d^{\nu+1}}{\nu!} \left( t_1 A_\nu(t_2) + (-1)^{\mu+\nu} t'_1 A_\nu(t'_2) \right) \\ &\quad D(j_1, j_2, \dots, j_n). \end{aligned} \quad (14)$$

We have already defined all the terms in Theorem 1.1. Here

$$g_n(d, m) = \sum_{j=1}^{d-1} \sin \frac{2\pi m j}{d} \cot^n(\frac{\pi j}{d} + \pi b). \quad (15)$$

**Proof :** Our contour will be the same as in Theorem 1.1. Let us take the complex function as

$$f(z) = \frac{e^{2\pi i m z} \cot^n(\pi z + \pi b)}{e^{2\pi i d z} - 1} + \frac{e^{-2\pi i m z} \cot^n(\pi z + \pi b)}{e^{-2\pi i d z} - 1} \quad (16)$$

and consider  $\frac{1}{2\pi i} \int_C f(z) dz$ . As before,  $\int_C f(z) dz = 0$ . The pole structure of this function is same as in Theorem 1.1. However, this time residue is zero at  $z = 0$ , so we take the sum from  $j = 1$ . Since  $\sin x$  vanishes at  $x = 0$ , we can take the sum from  $j = 0$ . The function  $f(z)$  has simple poles at  $z = \frac{j}{d}$ , with  $1 \leq j \leq d-1$ . The corresponding residues at these points are

$$\text{Res}(f, \frac{j}{d}) = \frac{1}{\pi d} \sin \frac{2\pi m j}{d} \cot^n(\frac{\pi j}{d} + \pi b). \quad (17)$$

The function  $f(z)$  also has a pole of order  $n$  at  $z = -b + 1$ . Using (4) and (5), as in the case of last theorem, we can obtain after a few steps of straight forward calculation,

$$\text{Res}(f, -b + 1) = \sum i^{\mu+\nu} 2^{\mu+\nu} \frac{m^\mu}{\mu!} \frac{d^\nu}{\nu!} \left( \frac{t_1}{\pi} A_\nu(t_2) + (-1)^{\mu+\nu} \frac{t'_1}{\pi} A_\nu(t'_2) \right) D(j_1, j_2, \dots, j_n). \quad (18)$$

Using (17), (18) and applying residue theorem we can easily obtain (14).

**Corollary 1.4 :** Let  $m$  and  $d$  integers be such that  $0 < m < d$ . Then

$$g_1(d, m) = -d \sin[(2m - d)b\pi] \text{cosec}(bd\pi). \quad (19)$$

**Proof :** Put  $n = 1$  in Theorem 1.3. Using the values  $A_0(t)$  and  $C_0$ , we can obtain this.

We will now consider sums involving sine and cosecants. We will do for even and odd powers of cosecants separately.

**Theorem 1.5 :** Let  $m, n$  and  $d$  denote positive integers with  $m < d$ . Then

$$h_n(d, m) = - \sum i^{\mu+\nu} 2^{\mu+\nu} \frac{m^\mu}{\mu!} \frac{d^{\nu+1}}{\nu!} \left( t_1 A_\nu(t_2) + (-1)^{\mu+\nu} t'_1 A_\nu(t'_2) \right) F(j_1, j_2, \dots, j_{2n}), \quad (20)$$

where the sum is over all nonnegative integers  $j_1, \dots, j_{2n}$ ,  $\mu$  and  $\nu$  such that  $2j_1 + \dots + 2j_{2n} + \mu + \nu = 2n - 1$  and  $\mu + \nu$  must be odd. All other terms have already been defined in Theorem 1.1 except for  $F(j_1, j_2, \dots, j_{2n})$ . Here

$$h_n(d, m) = \sum_{j=1}^{d-1} \sin \frac{2\pi m j}{d} \text{cosec}^{2n}(\frac{\pi j}{d} + \pi b). \quad (21)$$

The function  $F(j_1, j_2, \dots, j_{2n})$  is defined through the expansion

$$\text{cosec}(\pi z + \pi b) = \sum_{j=0}^{\infty} B_j \pi^{2j-1} (z - 1 + b)^{2j-1}, \quad (22)$$

where  $B_j$  is a polynomial of degree  $j$ . The first few terms are  $B_0 = -1$ ,  $B_1 = -\frac{1}{6}$ ,  $B_2 = -\frac{7}{360}$  and  $B_3 = -\frac{31}{15120}$ . Then

$$F(j_1, j_2, \dots, j_{2n}) = \prod_{r=1}^{2n} B_{j_r}. \quad (23)$$

**Proof :** Let's take the complex function as

$$f(z) = \frac{e^{2\pi i m z} \text{cosec}^{2n}(\pi z + \pi b)}{e^{2\pi i d z} - 1} + \frac{e^{-2\pi i m z} \text{cosec}^{2n}(\pi z + \pi b)}{e^{-2\pi i d z} - 1}. \quad (24)$$

We can use the same contour and follow the same procedure as in Theorem 1.1 and Theorem 1.3. The function  $f(z)$  has simple poles at  $z = \frac{j}{d}$ , with  $1 \leq j \leq d-1$  with residues

$$\text{Res}(f, \frac{j}{d}) = \frac{1}{\pi d} \sin \frac{2\pi m j}{d} \text{cosec}^{2n}(\frac{\pi j}{d} + \pi b). \quad (25)$$

The function  $f(z)$  also has a pole of order  $2n$  at  $z = -b + 1$ . Using (4) and (22) we can write

$$\begin{aligned}
f(z) = & t_1 \sum_{\mu=0}^{\infty} \frac{(2\pi i m)^{\mu}}{\mu!} (z+b-1)^{\mu} \left( \sum_{j=0}^{\infty} B_j \pi^{2j-1} (z-1+b)^{2j-1} \right)^{2n} \\
& \sum_{\nu=0}^{\infty} \frac{(2\pi i d)^{\nu}}{\nu!} A_{\nu}(t_2) (z+b-1)^{\nu} \\
& + t'_1 \sum_{\mu=0}^{\infty} \frac{(-2\pi i m)^{\mu}}{\mu!} (z+b-1)^{\mu} \left( \sum_{j=0}^{\infty} B_j \pi^{2j-1} (z-1+b)^{2j-1} \right)^{2n} \\
& \sum_{\nu=0}^{\infty} \frac{(-2\pi i d)^{\nu}}{\nu!} A_{\nu}(t'_2) (z+b-1)^{\nu}.
\end{aligned} \tag{26}$$

After a few steps of straightforward calculation, one obtains

$$\text{Res}(f, -b+1) = \sum i^{\mu+\nu} 2^{\mu+\nu} \frac{m^{\mu}}{\mu!} \frac{d^{\nu}}{\nu!} \left( \frac{t_1}{\pi} A_{\nu}(t_2) + (-1)^{\mu+\nu} \frac{t'_1}{\pi} A_{\nu}(t'_2) \right) F(j_1, j_2, \dots, j_n). \tag{27}$$

Using (25), (27) and applying residue theorem we can easily prove (20).

**Corollary 1.6 :** Let  $m$  and  $d$  be integers such that  $0 < m < d$ . Then

$$h_1(d, m) = d \operatorname{cosec}^2((b-1)d\pi) \left[ m \sin(2(b-1)(d-m)\pi) - (d-m) \sin(2(b-1)m\pi) \right]. \tag{28}$$

**Proof :** Put  $n = 1$  in Theorem 1.5. Using the values  $A_0(t)$ ,  $A_1(t)$  and  $B_0$ , one can show this.

We will next consider a sum involving cosine and even powers of the cosecant. We will have to only change the function slightly.

**Theorem 1.7 :** Let  $m, n$  and  $d$  denote positive integers with  $m < d$ . Then

$$\begin{aligned}
k_n(d, m) = & - \sum i^{\mu+\nu+1} 2^{\mu+\nu} \frac{m^{\mu}}{\mu!} \frac{d^{\nu+1}}{\nu!} \left( t_1 A_{\nu}(t_2) - (-1)^{\mu+\nu} t'_1 A_{\nu}(t'_2) \right) \\
& F(j_1, j_2, \dots, j_{2n}),
\end{aligned} \tag{29}$$

where the sum is over all nonnegative integers  $j_1, \dots, j_{2n}$ ,  $\mu$  and  $\nu$  such that  $2j_1 + \dots + 2j_{2n} + \mu + \nu = 2n - 1$  and  $\mu + \nu + 1$  must be even. Here

$$k_n(d, m) = \sum_{j=0}^{d-1} \cos \frac{2\pi m j}{d} \operatorname{cosec}^{2n} \left( \frac{\pi j}{d} + \pi b \right). \tag{30}$$

**Proof :** Let's take the complex function as

$$f(z) = \frac{e^{2\pi i m z} \operatorname{cosec}^{2n}(\pi z + \pi b)}{e^{2\pi i d z} - 1} - \frac{e^{-2\pi i m z} \operatorname{cosec}^{2n}(\pi z + \pi b)}{e^{-2\pi i d z} - 1}. \tag{31}$$

Then applying the same procedure as in Theorem 1.5, we can obtain (29).

**Corollary 1.8 :** Let  $m$  and  $d$  be integers such that  $0 < m < d$ . Then

$$k_1(d, m) = d \operatorname{cosec}^2((b-1)d\pi) \left[ m \cos(2(b-1)(d-m)\pi) + (d-m) \cos(2(b-1)m\pi) \right]. \tag{32}$$

**Proof :** Put  $n = 1$  in Theorem 1.7. Using the values  $A_0(t)$ ,  $A_1(t)$  and  $B_0$ , this result can be obtained.

Let us now consider odd powers of cosecants. In this case to keep the periodicity of the function  $f(z)$ , we shall have to use the argument of sine as  $\frac{\pi m j}{d}$ , instead of  $2\frac{\pi m j}{d}$  as earlier.

**Theorem 1.9 :** If  $m$  is an odd integer and  $n, d$  are positive integers with  $m < d$ . Then

$$l_n(d, m) = - \sum i^{\mu+\nu} 2^\nu \frac{m^\mu}{\mu!} \frac{d^{\nu+1}}{\nu!} \left( p_1 A_\nu(p_2) + (-1)^{\mu+\nu} p'_1 A_\nu(p'_2) \right) F(j_1, j_2, \dots, j_{2n-1}), \quad (33)$$

where the sum is over all nonnegative integers  $j_1, \dots, j_{2n-1}, \mu$  and  $\nu$  such that  $2j_1 + \dots + 2j_{2n-1} + \mu + \nu = 2n - 2$  and  $\mu + \nu$  must be even. Here  $p_1 = e^{-\pi i m(b-1)}$ ,  $p_2 = e^{-2\pi i d(b-1)}$ ,  $p'_1 = e^{\pi i m(b-1)}$  and  $p'_2 = e^{2\pi i d(b-1)}$

$$l_n(d, m) = \sum_{j=1}^{d-1} \sin \frac{\pi m j}{d} \operatorname{cosec}^{2n-1} \left( \frac{\pi j}{d} + \pi b \right). \quad (34)$$

**Proof :** Let's take the complex function as

$$f(z) = \frac{e^{\pi i m z} \operatorname{cosec}^{2n-1}(\pi z + \pi b)}{e^{2\pi i d z} - 1} + \frac{e^{-\pi i m z} \operatorname{cosec}^{2n-1}(\pi z + \pi b)}{e^{-2\pi i d z} - 1}. \quad (35)$$

Again proceeding in the same way as before we can obtain (33).

**Corollary 1.10 :** If  $m$  is an odd integer and  $n, d$  are positive integers with  $m < d$ , then

$$l_1(d, m) = -d \sin((m-d)b\pi) \operatorname{cosec}(\pi d b). \quad (36)$$

**Proof :** By putting  $n = 1$  in Theorem 1.9 and taking the values of  $A_0(p)$ , and  $B_0$ , one can easily see this.

Again as before, in the above sum sine can be replaced with cosine, by modifying  $f(z)$  slightly.

**Theorem 1.11 :** If  $m$  is an odd integer and  $n, d$  are positive integers with  $m < d$ , then

$$q_n(d, m) = - \sum i^{\mu+\nu+1} 2^\nu \frac{m^\mu}{\mu!} \frac{d^{\nu+1}}{\nu!} \left( p_1 A_\nu(p_2) - (-1)^{\mu+\nu} p'_1 A_\nu(p'_2) \right) F(j_1, j_2, \dots, j_{2n-1}), \quad (37)$$

where the sum is over all nonnegative integers  $j_1, \dots, j_{2n-1}, \mu$  and  $\nu$  such that  $2j_1 + \dots + 2j_{2n-1} + \mu + \nu = 2n - 2$  and  $\mu + \nu$  must be even. Here,

$$q_n(d, m) = \sum_{j=0}^{d-1} \cos \frac{\pi m j}{d} \operatorname{cosec}^{2n-1} \left( \frac{\pi j}{d} + \pi b \right). \quad (38)$$

**Proof :** Let's take the complex function as

$$f(z) = \frac{e^{\pi i m z} \operatorname{cosec}^{2n-1}(\pi z + \pi b)}{e^{2\pi i d z} - 1} - \frac{e^{-\pi i m z} \operatorname{cosec}^{2n-1}(\pi z + \pi b)}{e^{-2\pi i d z} - 1}. \quad (39)$$

Again proceeding in the same way as before we can obtain (37).

**Corollary 1.12 :** If  $m$  is an odd integer and  $n, d$  are positive integers with  $m < d$ , then

$$q_1(d, m) = d \cos((m-d)b\pi) \operatorname{cosec}(\pi d b). \quad (40)$$

**Proof :** By putting  $n = 1$  in Theorem 1.11 and taking the values of  $A_0(p)$ , and  $B_0$ , we can easily obtain this.

If we take tangent instead of cotangent in the sum of corollary 1.2, then we will get

$$\sum_{j=0}^{d-1} \cos \frac{2\pi m j}{d} \tan\left(\frac{\pi j}{d} + \pi b\right) = \begin{cases} (-1)^{m+1} d \cos[(2m-d)b\pi] \operatorname{cosec}(bd\pi); & \text{if } d = \text{even}, \\ (-1)^{m-d} d \sin[(2m-d)b\pi] \sec(bd\pi); & \text{if } d = \text{odd}. \end{cases} \quad (41)$$

Similarly, if in the sum of corollary 1.4, instead of cotangent if we take tangent then

$$\sum_{j=1}^{d-1} \sin \frac{2\pi m j}{d} \tan\left(\frac{\pi j}{d} + \pi b\right) = \begin{cases} (-1)^m d \sin[(2m-d)b\pi] \operatorname{cosec}(bd\pi); & \text{if } d = \text{even}, \\ (-1)^{m-d} d \cos[(2m-d)b\pi] \sec(bd\pi); & \text{if } d = \text{odd}. \end{cases} \quad (42)$$

For even  $d$  the sums in (41) and (42) are not defined for  $bd = \text{integer}$ , and for odd  $d$  the sums are not defined for  $bd = \frac{(\text{odd integer})}{2}$ .

Many more sums of the product of two trigonometric functions can be obtained with the same contour, with a change of functions. As an example, if we change slightly the arguments of the exponentials of the complex functions  $f(z)$ , we can get different kind of sums. To illustrate this, we can take the factor of 4 instead of 2 in the exponentials of the function  $f(z)$  in theorems 1.1 and 1.3. Following the same procedure as used there, we get

$$\sum_{j=0}^{2d-1} \cos \frac{2\pi m j}{d} \cot\left(\frac{\pi j}{2d} + \pi b\right) = 2d \cos[(2m-d)2b\pi] \operatorname{cosec}(2bd\pi) \quad (43)$$

and

$$\sum_{j=1}^{2d-1} \sin \frac{2\pi m j}{d} \cot\left(\frac{\pi j}{2d} + \pi b\right) = -2d \sin[(2m-d)2b\pi] \operatorname{cosec}(2bd\pi) \quad (44)$$

respectively.

These are just a sample of sums that one can obtain with the contour described in Theorem 1.1, but choosing a variety of integrands.

### 3 Products of More than Two Trigonometric Functions

We shall now consider a few sums involving the products of three trigonometric functions. We shall consider only those functions for which integral along the two sides of the contour cancel each other. Let us consider the following function

$$f(z) = \left[ \frac{e^{2\pi i m z}}{e^{2\pi i d z} - 1} - \frac{e^{-2\pi i m z}}{e^{-2\pi i d z} - 1} \right] \operatorname{cosec}(\pi z + \pi b_1) \cos(\pi z + \pi b_2). \quad (45)$$

Using the same contour and the same procedure as in Theorem 1 and 2, we get

$$\begin{aligned} & \sum_{j=0}^{d-1} \cos \frac{2\pi m j}{d} \operatorname{cosec}\left(\frac{\pi j}{d} + \pi b_1\right) \cos\left(\frac{\pi j}{d} + \pi b_2\right) \\ &= -d \cos[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \cos[\pi + (b_2 - b_1)\pi]. \end{aligned} \quad (46)$$

In the same way by replacing the last term by sine in (45), we get

$$\begin{aligned} & \sum_{j=0}^{d-1} \cos \frac{2\pi m j}{d} \operatorname{cosec}\left(\frac{\pi j}{d} + \pi b_1\right) \sin\left(\frac{\pi j}{d} + \pi b_2\right) \\ &= -d \cos[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \sin[\pi + (b_2 - b_1)\pi]. \end{aligned} \quad (47)$$

By replacing the cosecant in (45) by secant, we get following two sums

$$\sum_{j=0}^{d-1} \cos \frac{2\pi m j}{d} \sec\left(\frac{\pi j}{d} + \pi b_1\right) \cos\left(\frac{\pi j}{d} + \pi b_2\right) = \begin{cases} (-1)^{m+1} d \cos[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \\ \cos\left[\frac{\pi}{2} + (b_2 - b_1)\pi\right]; & \text{if } d = \text{even}, \\ (-1)^{m-d} d \sin[(2m-d)b_1\pi] \sec(b_1 d\pi) \\ \cos\left[\frac{\pi}{2} + (b_2 - b_1)\pi\right]; & \text{if } d = \text{odd}. \end{cases} \quad (48)$$

and

$$\sum_{j=0}^{d-1} \cos \frac{2\pi m j}{d} \sec\left(\frac{\pi j}{d} + \pi b_1\right) \sin\left(\frac{\pi j}{d} + \pi b_2\right) = \begin{cases} (-1)^{m+1} d \cos[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \\ \sin\left[\frac{\pi}{2} + (b_2 - b_1)\pi\right]; & \text{if } d = \text{even}, \\ (-1)^{m-d} d \sin[(2m-d)b_1\pi] \sec(b_1 d\pi) \\ \sin\left[\frac{\pi}{2} + (b_2 - b_1)\pi\right]; & \text{if } d = \text{odd}. \end{cases} \quad (49)$$

In the function (45), if we replace the minus sign between two exponentials by the plus sign, we will find

$$\begin{aligned} & \sum_{j=1}^{d-1} \sin \frac{2\pi m j}{d} \operatorname{cosec}\left(\frac{\pi j}{d} + \pi b_1\right) \cos\left(\frac{\pi j}{d} + \pi b_2\right) \\ &= d \sin[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \cos[\pi + (b_2 - b_1)\pi]. \end{aligned} \quad (50)$$

In the similar way replacing the last cosine term by sine, we get

$$\begin{aligned} & \sum_{j=1}^{d-1} \sin \frac{2\pi m j}{d} \operatorname{cosec}\left(\frac{\pi j}{d} + \pi b_1\right) \sin\left(\frac{\pi j}{d} + \pi b_2\right) \\ &= d \sin[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \sin[\pi + (b_2 - b_1)\pi]. \end{aligned} \quad (51)$$

Just like we obtained (48) and (49), we can find

$$\sum_{j=1}^{d-1} \sin \frac{2\pi m j}{d} \sec\left(\frac{\pi j}{d} + \pi b_1\right) \cos\left(\frac{\pi j}{d} + \pi b_2\right) = \begin{cases} (-1)^m d \sin[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \\ \cos\left[\frac{\pi}{2} + (b_2 - b_1)\pi\right]; & \text{if } d = \text{even}, \\ (-1)^{m-d} d \cos[(2m-d)b_1\pi] \sec(b_1 d\pi) \\ \cos\left[\frac{\pi}{2} + (b_2 - b_1)\pi\right]; & \text{if } d = \text{odd}. \end{cases} \quad (52)$$

and

$$\sum_{j=1}^{d-1} \sin \frac{2\pi m j}{d} \sec\left(\frac{\pi j}{d} + \pi b_1\right) \sin\left(\frac{\pi j}{d} + \pi b_2\right) = \begin{cases} (-1)^m d \sin[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \\ \sin\left[\frac{\pi}{2} + (b_2 - b_1)\pi\right]; & \text{if } d = \text{even}, \\ (-1)^{m-d} d \cos[(2m-d)b_1\pi] \sec(b_1 d\pi) \\ \sin\left[\frac{\pi}{2} + (b_2 - b_1)\pi\right]; & \text{if } d = \text{odd}. \end{cases} \quad (53)$$

Let us now take a different complex function. We can then obtain many more sums.

$$f(z) = \left[ \frac{e^{2\pi i m z}}{e^{2\pi i d z} - 1} - \frac{e^{-2\pi i m z}}{e^{-2\pi i d z} - 1} \right] \operatorname{cosec}(\pi z + \pi b_1) \operatorname{cosec}(\pi z + \pi b_2). \quad (54)$$

By using residue theorem, we can show that

$$\begin{aligned} & \sum_{j=0}^{d-1} \cos \frac{2\pi m j}{d} \operatorname{cosec}\left(\frac{\pi j}{d} + \pi b_1\right) \operatorname{cosec}\left(\frac{\pi j}{d} + \pi b_2\right) \\ &= -d \cos[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \operatorname{cosec}[\pi + (b_2 - b_1)\pi] \\ &\quad -d \cos[(2m-d)b_2\pi] \operatorname{cosec}(b_2 d\pi) \operatorname{cosec}[\pi + (b_1 - b_2)\pi]. \end{aligned} \quad (55)$$



In (54) instead of two cosecant, if we use one cosecant and one secant, we will get

$$\sum_{j=0}^{d-1} \cos \frac{2\pi m j}{d} \operatorname{cosec}\left(\frac{\pi j}{d} + \pi b_1\right) \sec\left(\frac{\pi j}{d} + \pi b_2\right) = \begin{cases} \left( -d \cos[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \sec[\pi + (b_2 - b_1)\pi] \right. \\ \left. + (-1)^{m+1} d \cos[(2m-d)b_2\pi] \operatorname{cosec}(b_2 d\pi) \right. \\ \left. \operatorname{cosec}\left[\frac{\pi}{2} + (b_1 - b_2)\pi\right] \right); & \text{if } d = \text{even}, \\ \left( -d \cos[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \sec[\pi + (b_2 - b_1)\pi] \right. \\ \left. + (-1)^{m-d} d \sin[(2m-d)b_2\pi] \sec(b_2 d\pi) \right. \\ \left. \operatorname{cosec}\left[\frac{\pi}{2} + (b_1 - b_2)\pi\right] \right); & \text{if } d = \text{odd}. \end{cases} \quad (56)$$

Again if both are secants, then

$$\sum_{j=0}^{d-1} \cos \frac{2\pi m j}{d} \sec\left(\frac{\pi j}{d} + \pi b_1\right) \sec\left(\frac{\pi j}{d} + \pi b_2\right) = \begin{cases} \left( (-1)^{m+1} d \cos[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \right. \\ \left. \sec\left[\frac{\pi}{2} + (b_2 - b_1)\pi\right] + (-1)^{m+1} d \cos[(2m-d)b_2\pi] \right. \\ \left. \operatorname{cosec}(b_2 d\pi) \sec\left[\frac{\pi}{2} + (b_1 - b_2)\pi\right] \right); & \text{if } d = \text{even}, \\ \left( (-1)^{m-d} d \sin[(2m-d)b_1\pi] \sec(b_1 d\pi) \right. \\ \left. \sec\left[\frac{\pi}{2} + (b_2 - b_1)\pi\right] + (-1)^{m-d} d \sin[(2m-d)b_2\pi] \right. \\ \left. \sec(b_2 d\pi) \sec\left[\frac{\pi}{2} + (b_1 - b_2)\pi\right] \right); & \text{if } d = \text{odd}. \end{cases} \quad (57)$$

Like (55), (56) and (57), we can get similar sums with sine as follows.

$$\begin{aligned} & \sum_{j=1}^{d-1} \sin \frac{2\pi m j}{d} \operatorname{cosec}\left(\frac{\pi j}{d} + \pi b_1\right) \operatorname{cosec}\left(\frac{\pi j}{d} + \pi b_2\right) \\ &= d \sin[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \operatorname{cosec}[\pi + (b_2 - b_1)\pi] \\ & \quad + d \sin[(2m-d)b_2\pi] \operatorname{cosec}(b_2 d\pi) \operatorname{cosec}[\pi + (b_1 - b_2)\pi]. \end{aligned} \quad (58)$$

$$\sum_{j=1}^{d-1} \sin \frac{2\pi m j}{d} \operatorname{cosec}\left(\frac{\pi j}{d} + \pi b_1\right) \sec\left(\frac{\pi j}{d} + \pi b_2\right) = \begin{cases} \left( d \sin[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \sec[\pi + (b_2 - b_1)\pi] + \right. \\ \left. (-1)^m d \sin[(2m-d)b_2\pi] \operatorname{cosec}(b_2 d\pi) \right. \\ \left. \operatorname{cosec}\left[\frac{\pi}{2} + (b_1 - b_2)\pi\right] \right); & \text{if } d = \text{even}, \\ \left( d \sin[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \sec[\pi + (b_2 - b_1)\pi] \right. \\ \left. + (-1)^{m-d} d \cos[(2m-d)b_2\pi] \sec(b_2 d\pi) \right. \\ \left. \operatorname{cosec}\left[\frac{\pi}{2} + (b_1 - b_2)\pi\right] \right); & \text{if } d = \text{odd}. \end{cases} \quad (59)$$

$$\sum_{j=0}^{d-1} \sin \frac{2\pi m j}{d} \sec\left(\frac{\pi j}{d} + \pi b_1\right) \sec\left(\frac{\pi j}{d} + \pi b_2\right) = \begin{cases} \left( (-1)^m d \sin[(2m-d)b_1\pi] \operatorname{cosec}(b_1 d\pi) \sec\left[\frac{\pi}{2} + (b_2 - b_1)\pi\right] \right. \\ \left. + (-1)^m d \sin[(2m-d)b_2\pi] \operatorname{cosec}(b_2 d\pi) \right. \\ \left. \sec\left[\frac{\pi}{2} + (b_1 - b_2)\pi\right] \right); & \text{if } d = \text{even}, \\ \left( (-1)^{m-d} d \cos[(2m-d)b_1\pi] \sec(b_1 d\pi) \sec\left[\frac{\pi}{2} + (b_2 - b_1)\pi\right] \right. \\ \left. + (-1)^{m-d} d \cos[(2m-d)b_2\pi] \sec(b_2 d\pi) \right. \\ \left. \sec\left[\frac{\pi}{2} + (b_1 - b_2)\pi\right] \right); & \text{if } d = \text{odd}. \end{cases} \quad (60)$$

In this section, we have considered the product of three trigonometric functions with simple powers. One can extend these results to higher powers of trigonometric functions, as well as products of more than three trigonometric functions. The list is endless. We have only illustrated a few cases.

## 4 Conclusion

We have obtained a number of finite sums involving products of two or more trigonometric functions. They were mostly based on a specific choice of contour and a wide variety of integrands. Many more sums can be obtained. For the simpler powers of trigonometric functions, these sums can be remarkably simple. Most of these sums involve tangents, cotangents, secants, and cosecants. For these functions, we don't have simple expansions for the sum and difference of variables in their arguments. Therefore, one has to compute them independently. We have calculated the sums for a few cases. One can extend these calculations in many different directions, as discussed in the above sections. Most of the simpler expressions should be in handbooks that have trigonometric sums.

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